

Edge number report 1: state of the art estimates for $n \leq 43$.

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Abstract

This first extracted report contains all lower and upper bounds for e-numbers $e(3, k; n)$, for $n \leq 43$, that I know. All but 24 of them are known (exactly). Very little of the proofs is given. A few consequences for upper classical Ramsey number bounds are mentioned.

1 Introduction.

Throughout the years, I have investigated e-numbers, and updated my tables of these and of properties for graphs with edge numbers close to the respective e-number. The results have been collected in the various updated versions of [1]. However, that work is not easily accessible; not only since I have not made it public, but since it is large, and based on a somewhat complex terminology, both for graph objects and for methods for dealing with them.

At present, I'm integrating the consequences of Goedgebeur's and Radziszowski's investigations in [4] into my tables. This is slow work; I have now more or less finished it up to vertex number 43. This has yielded a few improvements, compared both to [4] and to older versions of [1].

I have received some criticism for not making my results more accessible. In this report, I indeed try to present the more recent ones, as regards e-number bounds; but not the further Ramsey graph properties. I believe that this makes it easier to uaccess *the conclusions*; but it makes it harder to reproduce or improve *the proofs*. I outline a few proof examples; they may at least illustrate the 'Ramsey calculus' methods.

Moreover I also discuss upper bounds for e-numbers. This is an area not equally well covered by the literature, I think, and I'm not sure of how good the upper bounds I give here are, compared to the state-of-the-art.

Finally, the terminology is a bit experimentative. I try to make it more conformant to other recent state-of-the-art articles, and (against my instincts) leave a good bit undefined. I'll be very thankful for comments, both on this, and on the factual content of this report.

2 Definitions.

Throughout this work, all graphs $G = (V, E)$ are finite, simple, and undirected; and they are *triangle-free*; i. e., the clique number $\omega(G) \leq 2$.

The *second degree* of a vertex v in a graph G is

$$\deg^2(v) = \deg_G^2(v) := \sum_{w \in N(v)} \deg(w),$$

where $N(v)$ is the set of vertices adjacent to v . (The second degree is denoted $Z(v)$ in e. g. [4].) The induced G subgraph on $V \setminus (N(v) \cup \{v\})$ is denoted G_v .

G is an $(i, j; n, e)$ -graph and an $(i, j; n)$ -graph, if $\omega(G) < i$, its independence number $\alpha(G) < j$, $n(G) := |V| = n$, and $e(G) := |E| = e$.

For any positive integers i, j , and n , the *e-number* $e(i, j; n)$ is the minimal number e , such that there are $(i, j; n, e)$ -graphs, or ∞ , if no $(i, j; n)$ -graphs exist. They are of great interest for finding improved bounds of Ramsey numbers

$$R(i, j) := \min(n : e(i, j; n) = \infty),$$

but are also of interest in themselves.

In this report, we only discuss the e-numbers $e(3, j; n)$. For the estimates, we shall use a few linear or ‘piecewise linear’ functions on two integer variables, namely,

$$f_1(n, k) = \max(0, n - k, 3n - 5k, 5n - 10k, 6n - 13k);$$

$$f_2(n, k) = 8n - 19.5k;$$

$$f_3(n, k) = 9n - 23k; \text{ and}$$

$$f_4(n, k) = 6.8n - 15.6k.$$

Note, that $f_1(n, k) = 6n - 13k$, if $n \geq 3k$.

Occasionally, we mention the “linear graph invariant”

$$t(G) := e(G) - 6n(G) + 13\alpha(G).$$

$\mathcal{W}_{13;1,5}$ denotes the cyclic graph with 13 vertices (conventionally named u_1, \dots, u_{13}), and with two vertices forming an edge if the absolute value of their indices counted modulo 13 is either 1 or 5. (This graph very often is denoted H_{13} .)

For other concepts, background, et cetera, see the bibliography. In particular, we shall discuss some graphs given by means of *extension patterns*, which provide recipes for constructing them step-by-step; but neither the patterns and nor the corresponding graphs are formally described here.

3 Known general values.

For $n \leq 3.25k + 1.5$, all e-numbers are known. (This indeed includes all $e(3, k + 1; n)$ with $n \leq 43$ and $k \geq 13$.) To begin with, we have

Proposition 1. *For all positive integers n and k ,*

$$e(3, k + 1; n) \geq f_1(n, k).$$

The values are exact if and only if $n < R(3, k + 1)$, and moreover either $n \leq 3.25k - 1$, or $n = 3.25n$.

For a proof, see e. g. [10]. Note, that part of the result is the fact that $t(G) \geq 0$ for all (triangle-free) G .

Lemma 3.1. *Let k and n be positive integers, such that $3k \leq n < R(3, k + 1)$, but $e(3, k + 1; n) > f_1(n, k)$. Then $e(3, k + 1; n) = f_1(n, k) + 1 \iff -1 < n - 3.25k < 0$, $e(3, k + 1; n) = f_1(n, k) + 2 \iff 0 < n - 3.25k \leq 0.5$, and $e(3, k + 1; n) \geq f_1(n, k) + 3 \iff 0.5 < n - 3.25k$,*

The proof depends on deriving properties for graphs with $t(G) \leq 2$. In [1], indeed, all G with $t(G) \leq 1$ are characterised, and sufficient restrictions are found for those with $t(G) = 2$. (Actually, the complete characterising of the graphs with $t(G) = 0$ also is the main object of the stand-alone manuscript [2]. The $t(G) = 2$ result partly employs [4].)

Employing some constructions, we find that the lower bound in the last part of lemma 3.1 is exact in a few cases:

Lemma 3.2. *If $3k \leq n < R(3, k + 1)$ and $0.5 < n - 3.25k \leq 1.5$, then $e(3, k + 1; n) = f_1(n, k) + 3$.*

If $n > 3.25k + 1.5$, and moreover $k \leq 12$, then $e(3, k + 1; n) > f_1(n, k) + 3$; and I find it likely that this should hold also for all higher k . Moreover, I guess that

$$e(3, k + 1; n) \geq \max(f_2(n, k), f_3(n, k)), \tag{1}$$

too; but I am far from being able to prove this. The best general result I have for $n - 3.25k \gg 0$ is

Lemma 3.3. *For any n and k ,*

$$e(3, k + 1; n) \geq f_4(n, k).$$

(This is contained in [1, proposition 13.5], which is proved by means of a somewhat complicated induction argument).

4 The other values for $n \leq 34$.

For $n \leq 34$, all $e(3, k+1; n)$ are known. Actually, only 15 of them are ‘sporadic’, i. e., not given by the known Ramsey numbers, or in section 3; and they all have $n \geq 22$ and $6 \leq k \leq 9$. Thus, they are included in the following $e(3, l; n)$ table (where $l = k+1$):

$n \backslash l$	7	8	9	10
22	60	42	30	21
23	∞	49	35	25
24	∞	56	40	30
25	∞	65	46	35
26	∞	73	52	40
27	∞	85	61	45
28	∞	∞	68	51
29	∞	∞	77	58
30	∞	∞	86	66
31	∞	∞	95	73
32	∞	∞	104	81
33	∞	∞	118	90
34	∞	∞	129	99

Note, that all items under an ∞ in a column also are ∞ . In the sequel, in each column, just the top ∞ (if any) is printed.

5 The other values and estimates for $35 \leq n \leq 43$.

In the table, a single value indicates that this is the exact e-value. Two values separated by a dash (–) are the best known lower and upper bounds of the respective e-value. Again, $l = k+1$.

$n \backslash l$	9	10	11	12	13
35	140	107–108	84–85	68	55
36	∞	117–119	92–94	75	60
37		128–(132)	100–103	82	66
38		139–(143)	109–112	89–90	72
39		151–161	119–121	96–98	78
40		161– ∞	128–130	103–107	87
41		172– ∞	139–(150)	111–116	94
42		∞	149–(160)	120–125	101–102
43			159–(171)	129–134	108–111

The upper bounds within parentheses are rather preliminary; they are achieved by crude constructions, made more or less on the fly, since I am too ignorant to know where to look for the best actually achieved upper bounds. I expect there to have been constructions

or computer enumerations around for a while, giving better upper bounds for all five or most of them.

6 Consequences for Ramsey numbers.

By hand calculations or by means of e. g. the matlab programme FRANK ([6])¹, it is fairly easy to check for consequences for upper bounds on Ramsey numbers for any improvement of lower bounds of e-numbers. As compared to the combined values from [4] and older versions of [1], the sharper bounds presented here yield just two improved upper Ramsey number bounds.

It turned out that the improvement of the lower bound for $e(3, 12; 43)$ from 128 to 129 was crucial for deducing that

$$R(3, 19) \leq 132,$$

as reported in the latest dynamic survey on small Ramsey numbers ([8]).

The improvement of lower $e(3, 11; 39)$ bound from 117 ([4]) to 119 suffices to prove that

$$R(3, 16) \leq 97.$$

This bound is not (yet) included in the dynamic survey.

7 A few proof hints.

7.1 Lower bounds.

Most of the ‘sporadic’ lower bounds are found in [4]; and/or are direct consequences of lower bounds for smaller independence numbers. The exceptions are the lower bounds for $e(3, 11; 35)$, $e(3, 12; 38)$, $e(3, 12; 39)$, $e(3, 13; 41)$, $e(3, 13; 42)$, $e(3, 12; 43)$, $e(3, 11; 39)$, and $e(3, 11; 41)$.

The first six of these bounds, as well as the ‘general’ bounds, depend partly on theoretical classification of some ‘lower’ graphs, i. e., graphs with lower independence and vertex numbers; likewise, the two last ones depend on computational classification of some lower graphs. In all cases, there is some use of properties deduced for some lower graphs; and the general proof technique is to assume the existence of a graph G with ‘offendingly’ low $e(G)$, and then to deduce more and more precise conditions for G , until finally a contradiction is achieved. I’ll provide a few examples.

First, assume that G is a $(3, 11; 35)$ -graph with $e(G) \leq 83$; whence actually equality must hold. We then successively may prove:

¹The version of FRANK that I employ includes a test for raising the lower e-number bound in a few cases, where the only formally possible degree distributions all would have to contain either a triangle of low-degree vertices, or a low-degree vertex with too few low-degree neighbours (and thus a too high second degree). In practice, this only may happen, when the unraised e-number bound would be close to, but slightly less than, the e-value for some regular graph. This tweak yielded e. g. $e(3, 13; 51) \geq 179$.

- (a) $\delta(G) > 2$;
- (b) $\delta(G) > 3$;
- (c) any vertex of degree 4 has at most one neighbour of degree ≥ 5 ;
- (d) G_v has no $\mathcal{W}_{13;1,5}$ component for any vertex of degree 5; and
- (e) if $\deg(v) = 5$, then $\deg^2(v) \leq 24$.

Property (a) is immediate from the $e(3, 10; n)$ values.

(b) follows from (a), and from the fact that any $(3, 10; 31)$ -graph H with $e(H) \leq 74$ has $\delta(H) \geq 2$, strictly if $e(H) = 73$; and that there are at most two vertices of degree 2 in H , which (if indeed there are two of them) moreover must be adjacent.

(c) is immediate from (b), and the fact that $\deg^2(v) \leq 17$ for any vertex of degree 4.

(e) is an immediate consequence of (d), and of the fact that any $(3, 10; 29, 58)$ -graph does contain a $\mathcal{W}_{13;1,5}$ component. On the other hand, (e) directly yields a contradiction, since it means that we could calculate as if $e(3, 10; 29)$ were at least 59.

This just leaves the deduction of (d) from (b) and (c), which is somewhat less immediate. Assume for a contradiction that $\deg(v) = 5$, and that G_v has a $\mathcal{W}_{13;1,5}$ component. Let $N(v) = \{w_1, \dots, w_5\}$, and let U be the set of vertices in $\mathcal{W}_{13;1,5}$, which are not adjacent to any w_i ; in other words, $U = \{u \in V(\mathcal{W}_{13;1,5}) : \deg_G(u) = 4\}$.

Now, $|U| \leq 8$, since U cannot contain an independent 4-set; if it did, any edge between U and $N(v)$ would be redundant (in the sense that removing it from G would leave a graph which also did not contain an independent 11-set), but G can contain neither a redundant edge, nor a $\mathcal{W}_{13;1,5}$ component. Thus, and by inspection of $\mathcal{W}_{13;1,5}$, if U were non-empty, then there were a $u_j \in U$ with at most two neighbours in U , and therefore at least two neighbours of degrees ≥ 5 , contradicting (c).

Thus, instead, $U = \emptyset$; i. e., each vertex in $\mathcal{W}_{13;1,5}$ is adjacent to at least one w_i . This makes it possible to apply a “discharging” argument. ‘Charge’ each u_j with a unit charge, 1; and then ‘discharge’ each u_j by distributing its charge in equal proportions to its w_i neighbours. The total charge after discharging must stay 13. However, no w_i can receive a charge larger than 2.5; which means that $N(v)$ in total cannot carry a higher charge than 12.5. This is a contradiction; which indeed proves (d).

For a second example, assume that G is a $(3, 11; 41)$ -graph with $e(G) = 138$. There are few theoretic ways for such a graph to be ‘realised numerically’; in other words, if we let the degree distribution (degree sequence) of the graph be $(n_0, n_1, \dots, n_{10})$, then there are just a handful possible such sequences, for which the resulting Graver-Yackel defect $\gamma(G)$ would be non-negative (cf. [5] and [4]). In fact, also employing that a single vertex v of degree 8 would have $\deg^2(v) \leq 8 \cdot 7 = 56$, and thus a positive defect, and repressing all leading and trailing zeroes in the distributions, we would have one of

$$(11, 30), (12, 28, 1), (1, 9, 31), (2, 7, 32), \text{ and } (3, 5, 33)$$

as degree distribution, with the total defect $\gamma(G) = 3, 1, 2, 1$, and 0, respectively.

Put $F := \{v \in V : \deg(v) = 7 \text{ and } \deg^2(v) = 48\}$. In other words, F is the set of non-defect vertices of degree 7. Counting directly yields that $|F| \geq 27$, in each one of the cases.

For any $f \in F$, G_f is a $(3, 10; 33, 90)$ -graph. Now, Goedgebeur and Radziszowski classified all these graphs, and made a list of all 57099 of them available on the *House of Graphs* ([4]). Running the NAUTY ([7]) command `countg --Jd` on this list reveals that any such graph H contains an induced $K_{2,4}$, and has $\delta(H) \geq 4$. Moreover, a theoretical analysis shows that for any vertex v with $5 \leq \deg(v) \leq 7$, either $\delta(G_v) \geq 3$, or $\delta(G_v) = 2$ and $\gamma(v) = 3$, or $\gamma(v) > 3$.

Now, choose such an f ; if there is a vertex x of degree 8, actually choose $f \in F \cap N(x)$; choose a $K_{2,4} \subset V_f \subset V$, with $V(K_{2,4}) = \{a_1, a_2, b_1, \dots, b_4\}$ and $\deg(a_1) \leq \deg(a_2) \leq 7$, say. We now note, that

$$\delta(G_{a_i}) \leq \deg(a)_{3-i} - 4, \text{ for } i = 1, 2;$$

and employ this in estimating the defects of the a_i .

If $\deg(a_1) = 5$, then $\gamma(a_2) \geq 4 > 3 \geq \gamma(G)$, a contradiction. Likewise, if $\deg(a_1) = 6$, then $\gamma(a_2) = 3$, whence then $\gamma(a_1) = 0$; whence anyhow

$$6 \leq \deg(a_1) \leq 7 = \deg(a_2).$$

If $\deg(a_1) = 7$, then both a_1 and a_2 are defective, and the further defects in G sum up to at most 1, whence in particular then $\Delta(G) = 7$. Moreover, if $\deg(a_1) = 7$, then not both a_1 and a_2 may have defects ≥ 2 , whence instead then at least one of them has second valency 47, and thus at least five neighbours of degree 7, of which at least four belong to F . Thus, in this case, we may assume that $f' := b_4 \in F$; while if $\deg(a_1) = 6$, then let f' be arbitrarily chosen in $F \cap \text{lk } a_2$. In either case, there is some $K_{2,4}$ in $V_{f'}$, and this would also carry a defect at least 2, which would yield a total defect at least 4 in G , a contradiction.

7.2 Upper bounds.

For $n \leq 4k = 4l - 4n$ (but excepting $(n, l) \in \{(17, 6), (22, 7), (27, 8)\}$), there are constructions, whose connected components either are described by their extension patterns, or are one or the other of two *exceptional graphs*: The cyclic graph $\mathcal{W}_{13;1,5}$ (the unique $(3, 5; 13, 26)$ -graph), and the *twisted tesseract* (a $(3, 6; 16, 32)$ -graph). (The twisted tesseract also is denoted $(2\mathcal{W}_{8;1,4})_{5i}$ in [1]; i. e., it consists of two disjoint copies of $\mathcal{W}_{8;1,4}$, with the i 'th vertex in the first copy connected to the $5i$ 'th one in the second copy by an edge; where indices are taken modulo 8.)

The extension pattern of a graph G of the kind we consider here includes a triangle free graph T , such that

$$\begin{aligned} e(T) &\leq 2n(T), \\ \alpha(G) &= n(T), \\ n(G) &= 2n(T) + e(T), \text{ and} \\ e(G) &= n(T) + 2e(T) + \frac{1}{2} \sum_{x \in V(T)} \deg(x)^2. \end{aligned}$$

This yields that the graphs with only patterned and/or exceptional graphs as components indeed fulfil (1). In fact, for ‘most’ k and n with $3.25k \leq n \leq 4k$, we have such graphs realising equality in (1). However, there are some irregularities, for two reasons. First, each $\mathcal{W}_{13;1,5}$ component contributes 4 to the independence number of the graph; and there may not be an integer number of such components that realises equality in (1). Second, in general, for a connected patterned graph G with $3.25\alpha(G) \leq n(G) \leq 4\alpha(G)$, equality only can be achieved by having only vertices of degrees 3 and 4 in the pattern graph T (since other degree distributions yield higher $\sum_{V(T)} \deg(x)^2$); which for $(3, 10; 36)$ -graphs would force the pattern graph to be 4-regular, on 9 vertices. By inspection, there is no such triangle-free graph; the closest possible degree distribution is $(2, 5, 2)$ vertices of degrees $(3, 4, 5)$, respectively.

The upper bound 161 for $e(3, 10; 39)$ is reported by Goedgebeur and Radziszowski in [4], where it is noted that both they and Exoo have found huge amounts of $(3, 10; 39, 161)$ -graphs G , but no $(3, 10; 39)$ -graph with a lower number of edges.

For the five upper bounds within parentheses, let L be the regular (3^8) -type lace with constant offsets $(1, 3)$, a $(3, 9; 32, 104)$ -graph. (Laces are defined and investigated in [1]; they form a special class of patterned graphs.) Its family (v_1, \dots, v_8) of apices consists of non-adjacent vertices of degree 6, where moreover $\text{dist}(v_i, v_j) \geq 3$, if i and j have the same parity. The upper $e(3, 10; 37)$ ($e(3, 10; 38)$) bounds are achieved by a 4-extension (5-extension) of H , employing 3 (all 4) of the odd-indexed v_i , respectively; and the upper $e(3, 11; 41\text{—}43)$ bounds by making a further extension of one of these, employing the v_i with even indices.

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